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Periodic homogenisation of certain fully nonlinear partial differential equations

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Synopsis

We demonstrate how a fairly simple "perturbed test function" method establishes periodic homogenisation for certain Hamilton–Jacobi and fully nonlinear elliptic partial differential equations. The idea, following Lions, Papanicolaou and Varadhan, is to introduce (possibly nonsmooth) correctors, and to modify appropriately the theory of viscosity solutions, so as to eliminate then the effects of high-frequency oscillations in the coefficients.

1. Introduction

We investigate in this paper periodic homogenisation for certain fully nonlinear, first and second order partial differential equations (PDE). Our basic problem is this: given the PDE

$$F\left(D^2u^\varepsilon, Du^\varepsilon, u^\varepsilon, x, \frac{x}{\varepsilon}\right) = 0, \qquad (1.1)_\varepsilon$$

where the nonlinearity F is periodic in its last argument, we hope to show that the solutions u^ε converge somehow as ε goes to zero to a solution u of an effective limiting PDE, having the form

$$\bar{F}(D^2u, Du, u, x) = 0. \qquad (1.2)$$

The primary difficulties are both to discover the precise structure of $\bar{F}$ and also to justify rigorously the convergence.

We shall narrow our focus to those PDE verifying, formally at least, a maximum principle, so that we may invoke the theory of the weak or so-called viscosity solutions. This approach interprets a solution of $(1.1)_\varepsilon$ or (1.2) in terms of its pointwise behaviour with respect to a smooth test function ϕ (see [**7, 6, 16, 13, 10**], etc.).

Such a formulation turns out to be extremely useful for homogenisation questions, where both the bad nonlinearity and the rapid periodic oscillations preclude any very good uniform estimates on $\{u^\varepsilon\}_{\varepsilon>0}$. Indeed, if we employ the standard methods of asymptotic expansions to seek a representation

$$u^\varepsilon = u + \varepsilon u_1 + \varepsilon^2 u_2 + \dots, \qquad (1.3)$$

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we find formally that both the limit u and the correctors $u_1, u_2, \ldots$ must necessarily satisfy various bad nonlinear PDE, which in general do not have smooth solutions. On the other hand, the earlier paper **[9]** proposed replacing (1.3) with a corresponding expansion

$$\phi^\varepsilon = \phi + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \ldots, \tag{1.4}$$

the smooth test function ϕ being given. The general hope is that since ϕ is regular, the appropriate correctors $\phi_1, \phi_2, \ldots$ will be smooth as well. This turns out to be the case for the various quasilinear PDE analysed in **[9]**. For the situation at hand, however, the full nonlinearity forces $\phi_1, \phi_2, \ldots$ and thus ϕ^ε to be nonsmooth, and consequently the techniques of **[9]** are not applicable. But we demonstrate instead here that nevertheless we can modify viscosity solution techniques to handle convergence in various situations, in spite of the nonsmoothness of our perturbed test functions ϕ^ε. In this endeavour, we shall employ a number of ideas from the unpublished paper of Lions, Papanicolaou and Varadhan **[15]** (which predates **[9]**).

In Section 2 we discuss the general theory of homogenisation of certain first order fully nonlinear PDE and in Section 3, second order fully nonlinear elliptic equations. Section 4 investigates the passage from second order to first order PDE, in the presence of rapid oscillation. Finally, Section 5 provides several simple examples for which more-or-less explicit formulae can be had for the effective limiting PDE.

For more information concerning homogenisation of linear and quasilinear PDE, consult **[17, 2, 3]** and the references therein.

I am very grateful to P.-L. Lions for many useful suggestions, and especially for providing me with a copy of the unpublished paper **[15]**, several key results of which I reproduce below for the reader's convenience.

2. First-order equations

To illustrate most clearly the basic techniques, let us consider first the model problem

$$\begin{cases} H\left(Du^\varepsilon, u^\varepsilon, x, \dfrac{x}{\varepsilon}\right) = 0 & \text{in } \Omega, \\ u^\varepsilon = 0 & \text{in } \partial\Omega, \end{cases} \tag*{(2.1)$_\varepsilon$}$$

where Ω denotes a bounded smooth open subset of $\mathbb{R}^n$, and

$$H: \mathbb{R}^n \times \mathbb{R} \times \bar{\Omega} \times \mathbb{R}^n \to \mathbb{R}$$

is a given smooth function. Our primary hypothesis is that

$$\text{the mapping } y \mapsto H(p, u, x, y) \text{ is } Y\text{-periodic} \tag{2.2}$$

for all p, u, x, where $Y \equiv [0, 1]^n$, the unit cube in $\mathbb{R}^n$. We additionally require

$$\lim_{|p| \to \infty} H(p, u, x, y) = +\infty, \quad \text{uniformly on } B(0, L) \times \bar{\Omega} \times \mathbb{R}^n \text{ for each } L > 0, \tag{2.3}$$

$$u \mapsto H(p, u, x, y) - \mu u \text{ is nondecreasing} \tag{2.4}$$

for some $\mu>0$ and all p, x, y, and

$$\begin{cases} H \text{ is Lipschitz continuous on} \\ B(0, L)\times B(0, L)\times\bar\Omega\times\mathbb{R}^n \text{ for each } L>0. \end{cases} \tag{2.5}$$

Suppose finally for each $\varepsilon>0$, $u^\varepsilon\in C(\bar\Omega)$ is a viscosity solution of $(2.1)_\varepsilon$. In view of hypotheses (2.3), (2.4) and standard theory for viscosity solutions,

$$\sup_{0<\varepsilon<1}\|u^\varepsilon\|_{C^{0,1}(\bar\Omega)}<\infty; \tag{2.6}$$

and thus we may extract a subsequence $\{u^{\varepsilon_j}\}_{j=1}^\infty\subset\{u^\varepsilon\}_{\varepsilon>0}$ and a function $u\in C^{0,1}(\bar\Omega)$ so that

$$u^{\varepsilon_j}\to u \quad \text{uniformly on } \bar\Omega \text{ as } \quad \varepsilon_j\to 0. \tag{2.7}$$

We propose to find a nonlinear first order PDE which u solves in the viscosity sense. The key insight is due to Lions, Papanicolaou and Varadhan **[15]**:

LEMMA 2.1. *For each fixed $p\in\mathbb{R}^n$, $u\in\mathbb{R}$ and $x\in\bar\Omega$, there exists a unique real number λ for which the PDE*

$$\begin{cases} H(D_yv+p, u, x, y)=\lambda & \textit{in } \mathbb{R}^n \\ v \quad Y\text{-periodic} \end{cases} \tag{2.8}$$

has a viscosity solution $v\in C^{0,1}(\mathbb{R}^n)$.

Proof (**[15]**). 1. Given $p\in\mathbb{R}^n$, $u\in\mathbb{R}$ and $x\in\bar\Omega$, we consider for each $\delta>0$ the approximating PDE

$$\delta w^\delta+H(D_yw^\delta+p, u, x, y)=0 \quad \text{in } \mathbb{R}^n. \tag*{$(2.9)_\delta$}$$

Owing to the hypotheses on H, there exists a unique Lipschitz viscosity solution w^δ of $(2.9)_\delta$, and in particular the uniqueness implies w^δ to be Y-periodic. We have in addition the estimate

$$\sup_{0<\delta<1}\|\delta w^\delta\|_{L^\infty(Y)}\leqq\|H(p, u, x, \cdot)\|_{L^\infty(Y)}<\infty; \tag{2.10}$$

and thus from (2.3)

$$\sup_{0<\delta<1}\|Dw^\delta\|_{L^\infty(Y)}<\infty. \tag{2.11}$$

Now set

$$v^\delta\equiv w^\delta-\min_Y w^\delta. \tag{2.12}$$

Then v^δ is Y-periodic for each $0<\delta<1$ and

$$\sup_{0<\delta<1}\|v^\delta\|_{C^{0,1}(Y)}<\infty. \tag{2.13}$$

Utilising (2.10), (2.13), we extract a subsequence $\{(v^{\delta_j}, \delta_jw^{\delta_j})\}_{j=1}^\infty\subset\{(v^\delta, \delta w^\delta)\}_{0<\delta<1}$ so that

$$\begin{cases} \delta_j\to 0, & \\ v^{\delta_j}\to v & \text{uniformly in } \mathbb{R}^n, \\ \delta_jw^{\delta_j}\to-\lambda & \text{uniformly in } \mathbb{R}^n, \end{cases} \tag{2.14}$$

for some function $v \in C^{0,1}(\mathbb{R}^n)$, v Y-periodic, and some constant $\lambda \in \mathbb{R}$. Passing to limits in the viscosity sense in $(2.9)_\delta$, we deduce that the pair (v, λ) solves (2.8).

2. Uniqueness of λ follows from the comparison theorem for viscosity solutions. Indeed, suppose $(\hat{v}, \hat{\lambda})$ solves

$$\begin{cases} H(D_y\hat{v} + p, u, x, y) = \hat{\lambda} & \text{in } \mathbb{R}^n, \\ \hat{v} \quad Y\text{-periodic}, \end{cases} \tag{2.15}$$

and, say, $\hat{\lambda} > \lambda$. Adding a constant to v if necessary, we may suppose as well that

$$v > \hat{v} \quad \text{in } \mathbb{R}^n. \tag{2.16}$$

But for $\varepsilon > 0$ small enough,

$$\varepsilon\hat{v} + H(D_y\hat{v} + p, u, x, y) \geqq \varepsilon v + H(D_y v + p, u, x, y) \quad \text{in } \mathbb{R}^n$$

in the viscosity sense; whence

$$\hat{v} \geqq v \quad \text{in } \mathbb{R}^n,$$

a contradiction to (2.16). □

To display explicitly the dependence of λ on p, u and x, let us hereafter write

$$\lambda \equiv \bar{H}(p, u, x) \quad (p \in \mathbb{R}^n, u \in \mathbb{R}, x \in \bar{\Omega}), \tag{2.17}$$

defining thereby the *effective Hamiltonian* $\bar{H}$. Thus the *cell problem* (2.8) reads

$$\begin{cases} H(D_y v + p, u, x, y) = \bar{H}(p, u, x) & \text{in } \mathbb{R}^n, \\ v \quad Y\text{-periodic}. \end{cases} \tag{2.18}$$

Following **[15]**, we record next various properties of $\bar{H}$ inherited from H.

LEMMA 2.2. (a) $\lim_{|p| \to \infty} \bar{H}(p, u, x) = +\infty$, *uniformly on* $B(0, L) \times \bar{\Omega}$ *for each* $L > 0$.

(b) *The mapping* $u \mapsto \bar{H}(p, u, x) - \mu u$ *is nondecreasing for each* p, x.

(c) $\bar{H}$ *is Lipschitz continuous on* $B(0, L) \times B(0, L) \times \bar{\Omega}$ *for each* $L > 0$.

(d) *If* H *is convex in* p, *so is* $\bar{H}$.

Various additional properties of $\bar{H}$ may be found in **[15]**.

Proof. 1. Fix $M > 0$. Evaluating $(2.9)_\delta$ at a point $y_0 \in Y$ where w^δ attains its maximum, we discover

$$\delta w^\delta(y_0) + H(p, u, x, y_0) \leqq 0.$$

Thus hypothesis (2.3) ensures

$$-\delta w^\delta(y_0) \geqq M$$

provided $|p|$ is large enough. Letting δ tend to zero and recalling (2.14), (2.17), we find that

$$\bar{H}(p, u, x) \geqq M$$

if $|p|$ is sufficiently large. This verifies assertion (a).

2. Suppose $\hat{u} \geqq u$, $\delta > 0$, w^δ solves $(2.9)_\delta$ and $\hat{w}^\delta$ solves

$$\delta \hat{w}^\delta + H(D_y \hat{w}^\delta + p, \hat{u}, x, y) = 0 \quad \text{in } \mathbb{R}^n. \tag{$2.19)_\delta$}$$

In view of hypothesis (2.4) and standard comparison principles for viscosity solutions, we find

$$-\delta \hat{w}^\delta \geqq -\delta w^\delta + \mu(\hat{u} - u) \quad \text{in } \mathbb{R}^n;$$

and so, upon sending δ to zero, we deduce

$$\bar{H}(p, \hat{u}, x) \geqslant \bar{H}(p, u, x) + \mu(\hat{u} - u).$$

Assertion (b) is proved.

3. Choose $p, \hat{p} \in \mathbb{R}^n$, $u, \hat{u} \in \mathbb{R}$, $x, \hat{x} \in \bar{\Omega}$ with $|p|, |\hat{p}|, |u|, |\hat{u}| \leqslant L$. Let w^δ solve $(2.9)_\delta$ and $\hat{w}^\delta$ solve

$$\delta \hat{w}^\delta + H(D_y \hat{w}^\delta + \hat{p}, \hat{u}, \hat{x}, y) = 0 \quad \text{in } \mathbb{R}^n. \tag{$2.20)_\delta$}$$

Now in view of (2.5)

$$H(q + \hat{p}, \hat{u}, \hat{x}, y) \leqq H(q + p, u, x, y) + C(|p - \hat{p}| + |u - \hat{u}| + |x - \hat{x}|)$$

for all q, y with $|q| \leqq L'$ and a constant $C = C(L, L')$. Thus

$$\delta \hat{w}^\delta + H(D_y \hat{w}^\delta + p, u, x, y) \geqq -C(|p - \hat{p}| + |u - \hat{u}| + |x - \hat{x}|) \quad \text{in } \mathbb{R}^n$$

in the viscosity sense. This PDE and $(2.9)_\delta$ then imply

$$\delta w^\delta - \delta \hat{w}^\delta \leqq C(|p - \hat{p}| + |u - \hat{u}| + |x - \hat{x}|) \quad \text{in } \mathbb{R}^n,$$

and so in the limit

$$\bar{H}(\hat{p}, \hat{u}, \hat{x}) - \bar{H}(p, u, x) \leqq C(|p - \hat{p}| + |u - \hat{u}| + |x - \hat{x}|).$$

A similar argument implies as well that

$$\bar{H}(\hat{p}, \hat{u}, \hat{x}) - \bar{H}(p, u, x) \geqq -C(|p - \hat{p}| + |u - \hat{u}| + |x - \hat{x}|).$$

4. Assume now that H is convex in p. Fix $p, q \in \mathbb{R}^n$, $u \in \mathbb{R}$, $x \in \Omega$, and let v^p, v^q, $v^{(p+q)/2}$ be Y-periodic viscosity solutions of the PDE

$$\begin{cases} H(D_y v^p + p, u, x, y) = \bar{H}(p, u, x), \\ H(D_y v^q + q, u, x, y) = \bar{H}(q, u, x) \quad \text{in } \mathbb{R}^n, \\ H\left(D_y v^{(p+q)/2} + \dfrac{p+q}{2}, u, x, y\right) = \bar{H}\left(\dfrac{p+q}{2}, u, x\right). \end{cases} \tag{2.21}$$

Subtracting a constant from $v^{(p+q)/2}$ if needs be, we may assume

$$v^{(p+q)/2} < \tfrac{1}{2} v^p + \tfrac{1}{2} v^q \quad \text{in } \mathbb{R}^n. \tag{2.22}$$

For later contradiction, let us suppose that

$$\bar{H}\left(\frac{p+q}{2}, u, x\right) > \tfrac{1}{2}\bar{H}(p, u, x) + \tfrac{1}{2}\bar{H}(q, u, x). \tag{2.23}$$

We now *claim*

$$H\left(D_y\left(\frac{v^p+v^q}{2}\right)+\frac{p+q}{2}, u, x, y\right) \leq \tfrac{1}{2}\bar{H}(p, u, x)+\tfrac{1}{2}\bar{H}(q, u, x) \quad \text{in } \mathbb{R}^n \tag{2.24}$$

in the viscosity sense. To see this, set

$$w \equiv \frac{v^p+v^q}{2}$$

and write

$$w_\varepsilon = \eta_\varepsilon^* w,$$

η_ε denoting the usual mollifier with support in the ball $B(0, \varepsilon)$. Then

$$\begin{aligned} H\left(D_y w_\varepsilon+\frac{p+q}{2}, u, x, y\right) &\leqq \int_{B(y,\varepsilon)} \eta_\varepsilon(y-z) H\left(D_y w(z)+\frac{p+q}{2}, u, x, y\right) dz \\ &= \int_{B(y,\varepsilon)} \eta_\varepsilon(y-z) H\left(D_y w(z)+\frac{p+q}{2}, u, x, z\right) dz + o(1) \\ &\leqq \frac{1}{2}\int_{B(y,\varepsilon)} \eta_\varepsilon(y-z) H(D_y v^p(z)+p, u, x, z)\, dz \\ &\quad + \frac{1}{2}\int_{B(y,\varepsilon)} \eta_\varepsilon(y-z) H(D_y v^q(z)+q, u, x, z)\, dz + o(1) \\ &= \tfrac{1}{2}\bar{H}(p, u, x)+\tfrac{1}{2}\bar{H}(q, u, x)+o(1) \end{aligned}$$

as $\varepsilon \to 0$. The last equality is valid since the Lipschitz functions v^p and v^q solve their PDE almost everywhere. Sending ε to zero, we verify (2.24).

Utilising now (2.23), (2.24), we deduce

$$\frac{v^p+v^q}{2} \leqq v^{(p+q)/2} \quad \text{in } \mathbb{R}^n,$$

in contradiction to (2.22). □

We at last verify that the effective Hamiltonian $\bar{H}$ determines the limit PDE for our rapidly oscillating problems $(2.1)_\varepsilon$.

THEOREM 2.3. *Assume H verifies* (2.2)–(2.5). *Then u is the unique viscosity solution of*

$$\begin{cases} \bar{H}(Du, u, x)=0 & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega. \end{cases} \tag{2.25}$$

Proof. 1. Fix $\phi \in C^\infty(\Omega)$ and suppose first that $u-\phi$ has a *strict* local maximum at a point $x_0 \in \Omega$, with

$$u(x_0)=\phi(x_0). \tag{2.26}$$

We must show that

$$\bar{H}(D\phi(x_0), \phi(x_0), x_0) \leq 0. \tag{2.27}$$

Let us thus for later contradiction assume to the contrary that

$$\bar{H}(D\phi(x_0),\ \phi(x_0),\ x_0) \equiv \theta > 0. \tag{2.28}$$

Now set $p = D\phi(x_0)$, $u = \phi(x_0)$, $x = x_0$ in (2.18) and choose $v \in C^{0,1}(\mathbb{R}^n)$ to be a viscosity solution of

$$\begin{cases} H(D_y v + D\phi(x_0),\ \phi(x_0),\ x_0,\ y) = \bar{H}(D\phi(x_0),\ \phi(x_0),\ x_0) = \theta & \text{in } \mathbb{R}^n, \\ v \quad Y\text{-periodic}. \end{cases} \tag{2.29}$$

Define then the perturbed test function

$$\phi^\varepsilon(x) \equiv \phi(x) + \varepsilon v\left(\frac{x}{\varepsilon}\right) \quad (x \in \bar{\Omega}).$$

Notice that ϕ^ε is Lipschitz continuous, but is not C^1 in general.

2. We now *claim* that

$$H\left(D\phi^\varepsilon(x),\ \phi^\varepsilon(x),\ x,\ \frac{x}{\varepsilon}\right) \geqq \frac{\theta}{2} \quad \text{in } B(x_0, r) \tag{2.30$_\varepsilon$}$$

in the viscosity sense, for some sufficiently small radius $r > 0$ to be selected below. (Here $B(x_0, r)$ denotes the open ball with centre x_0, radius r.) To see this, choose any $\psi \in C^\infty(\mathbb{R}^n)$ and suppose $\phi^\varepsilon - \psi$ has a minimum at a point $x_1 \in B(x_0, r)$, with

$$\phi^\varepsilon(x_1) = \psi(x_1). \tag{2.31}$$

Then the mapping

$$x \mapsto \varepsilon v\left(\frac{x}{\varepsilon}\right) - [\psi(x) - \phi(x)]$$

has a minimum at x_1, and so

$$y \mapsto v(y) - \eta(y)$$

has a minimum at $y_1 \equiv x_1/\varepsilon$, for

$$\eta(y) \equiv \frac{1}{\varepsilon}(\psi(\varepsilon y) - \phi(\varepsilon y)). \tag{2.32}$$

Since v is a viscosity solution of (2.29), we deduce

$$H(D\eta(y_1) + D\phi(x_0),\ \phi(x_0),\ x_0,\ y_1) \geqq \theta.$$

Therefore (2.32) gives

$$H\left(D\psi(x_1) + D\phi(x_0) - D\phi(x_1),\ \phi(x_0),\ x_0,\ \frac{x_1}{\varepsilon}\right) \geqq \theta.$$

Consequently

$$H\left(D\psi(x_1),\ \psi(x_1),\ x_1,\ \frac{x_1}{\varepsilon}\right) \geqslant \frac{\theta}{2}$$

if $r > 0$ is small enough. We have verified (2.30)$_\varepsilon$.

3. In view of $(2.1)_\varepsilon$, $(2.30)_\varepsilon$, and standard comparison results, we conclude

$$(u^\varepsilon - \phi^\varepsilon)(x_0) \leq \max_{\partial B(x_0,r)} (u^\varepsilon - \phi^\varepsilon).$$

Now send $\varepsilon \to 0$ to find

$$(u - \phi)(x_0) \leqq \max_{\partial B(x_0,r)} (u - \phi),$$

a contradiction to the assertion that $u - \phi$ has a *strict* local maximum at x_0. Thus (2.28) is untenable and so (2.27) is proved.

4. The opposite inequality similarly obtains should $u - \phi$ have a strict local minimum at a point $x_0 \in \Omega$. Consequently u is a (and, by uniqueness, the) viscosity solution of (2.25). $\square$

3. Second-order elliptic equations

We next study periodic homogenisation for fully nonlinear elliptic PDE having the form

$$\begin{cases} F\left(D^2u^\varepsilon, Du^\varepsilon, u^\varepsilon, x, \dfrac{x}{\varepsilon}\right) = 0 & \text{in } \Omega, \\ u^\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1$_\varepsilon$}$$

where

$$F : S^{n\times n} \times \mathbb{R}^n \times \mathbb{R} \times \bar{\Omega} \times \mathbb{R}^n \to \mathbb{R}$$

is a given smooth function, $S^{n\times n}$ denoting the space of real, symmetric $n \times n$ matrices. We suppose

$$\text{the mapping } y \mapsto F(R, p, u, x, y) \text{ is } Y\text{-periodic} \tag{3.2}$$

for all R, p, u, x, and make the additional uniform ellipticity assumption

$$\begin{cases} \text{there exists a constant } \theta \text{ such that} \\ \theta\,|\xi|^2 \leqq -\dfrac{\partial F}{\partial r_{ij}}(R, p, u, x, y)\xi_i\xi_j \quad (\xi \in \mathbb{R}^n) \end{cases} \tag{3.3}$$

for all R, p, u, x, y. Let us hypothesise as well

$$u \mapsto F(R, p, u, x, y) - \mu u \text{ is nondecreasing} \tag{3.4}$$

for some $\mu > 0$ and all R, p, x, y,

$$F \text{ is Lipschitz on } S^{n\times n} \times \mathbb{R}^n \times \mathbb{R} \times \bar{\Omega} \times \mathbb{R}^n. \tag{3.5}$$

Assume now for each $\varepsilon > 0$ that $u^\varepsilon \in C(\bar{\Omega})$ is a viscosity solution of $(3.1)_\varepsilon$. Owing to (3.3) and estimates of Krylov and Safonov, extended to viscosity solutions of elliptic PDE by Trudinger [**18**] and Caffarelli [**5**], there exists $\gamma > 0$ for which

$$\sup_{0<\varepsilon<1} \|u^\varepsilon\|_{C^{0,\gamma}(\bar{\Omega})} < \infty. \tag{3.6}$$

As a consequence we may extract a subsequence $\{u^{\varepsilon_j}\}_{j=1}^{\infty} \subset \{u^{\varepsilon}\}_{\varepsilon>0}$ and a function $u \in C^{0,\gamma}(\bar{\Omega})$ with

$$u^{\varepsilon_j} \to u \quad \text{uniformly on } \bar{\Omega}. \tag{3.7}$$

As before, we wish to ascertain a nonlinear elliptic PDE which u solves in the viscosity sense.

LEMMA 3.1. *For each fixed $R \in S^{n\times n}$, $p \in \mathbb{R}^n$, $u \in \mathbb{R}$ and $x \in \bar{\Omega}$ there exists a unique real number λ for which the PDE*

$$\begin{cases} F(D_y^2 v + R, p, u, x, y) = \lambda & \text{in } \mathbb{R}^n \\ v \quad Y\text{-periodic} \end{cases} \tag{3.8}$$

has a viscosity solution $v \in C^{1,\gamma}(\mathbb{R}^n)$, for some $\gamma > 0$.

Proof. We mimic the proof of Lemma 2.1 by considering for $\delta > 0$ the approximating problem

$$\delta w^{\delta} + F(D_y^2 w^{\delta} + R, p, u, x, y) = 0 \quad \text{in } \mathbb{R}^n. \tag{3.9$_\delta$}$$

This PDE has a unique bounded viscosity solution w^{δ} (see, for instance, **[10, 19]**), which, owing to the uniqueness and hypothesis (3.2), is Y-periodic. Additionally

$$\sup_{0<\delta<1} \|\delta w^{\delta}\|_{L^{\infty}(Y)} \leq \|F(R, p, u, x, \cdot)\|_{L^{\infty}(Y)} < \infty. \tag{3.10}$$

In addition, the Krylov–Safonov estimates assert

$$\sup_{0<\delta<1} \|w^{\delta}\|_{C^{0,\gamma}(Y)} < \infty \tag{3.11}$$

for some $\gamma > 0$.

We now proceed as in the proof of Lemma 2.1. The $C^{1,\gamma}$ regularity follows from Trudinger **[19]**. $\square$

We write

$$\lambda = \bar{F}(R, p, u, x) \tag{3.12}$$

to exhibit explicitly the dependence of λ on R, p, u, x. Then (3.8) reads

$$\begin{cases} F(D_y^2 v + R, p, u, x, y) = \bar{F}(R, p, u, x) & \text{in } \mathbb{R}^n, \\ v \quad Y\text{-periodic}. \end{cases} \tag{3.13}$$

LEMMA 3.2. (a) *$\bar{F}$ is uniformly elliptic in the sense that*

$$\bar{F}(R+S, p, u, x) + \theta \operatorname{tr}(S) \leqq \bar{F}(R, p, u, x) \quad \text{if } S \geqq 0 \tag{3.14}$$

for all R, p, u, x.

(b) *The mapping $u \mapsto \bar{F}(R, p, u, x) - \mu u$ is nondecreasing for all R, p, x.*
(c) *$\bar{F}$ is Lipschitz on $S^{n\times n} \times \mathbb{R}^n \times \mathbb{R} \times \bar{\Omega}$.*
(d) *If F is convex in R, so is $\bar{F}$.*

Proof. 1. Assume $S \geqq 0$, and let v^R, v^{R+S} be Y-periodic viscosity solutions of

$$\begin{cases} F(D_y^2 v^R + R, p, u, x, y) = \bar{F}(R, p, u, x) \\ F(D_y^2 v^{R+S} + R + S, p, u, x, y) = \bar{F}(R+S, p, u, x) \end{cases} \quad \text{in } \mathbb{R}^n. \tag{3.15}$$

We may as well suppose additionally

$$v^{R+S} < v^R. \tag{3.16}$$

Assume then for later contradiction that

$$\bar{F}(R+S, p, u, x) > \bar{F}(R, p, u, x) - \theta \operatorname{tr}(S). \tag{3.17}$$

We now *claim* that

$$F(D_y^2 v^{R+S} + R, p, u, x, y) \geq \bar{F}(R, p, u, x) \quad \text{in } \mathbb{R}^n \tag{3.18}$$

in the viscosity sense. To see this, let $\phi \in C^\infty(\mathbb{R}^n)$ and suppose $v^{R+S} - \phi$ has a local minimum at a point $y_0 \in \mathbb{R}^n$. In view of (3.15) we have

$$F(D^2\phi(y_0) + R + S, p, u, x, y_0) \geqq \bar{F}(R+S, p, u, x).$$

Consequently (3.3) yields

$$\begin{aligned} F(D^2\phi(y_0) + R, p, u, x, y_0) &\geqq F(D^2\phi(y_0) + R + S, p, u, x_0) + \theta \operatorname{tr}(S) \\ &\geqq \bar{F}(R+S, p, u, x) + \theta \operatorname{tr}(S) \\ &> \bar{F}(R, p, u, x) \quad \text{by (3.17).} \end{aligned}$$

This establishes (3.18).

Owing now to (3.15), (3.18) and comparison theorems for viscosity solutions, we discover

$$v^{R+S} \geqq v^R,$$

a contradiction to (3.16). This verifies assertion (a).

2. The proofs of assertions (b) and (c) are similar to the corresponding proofs for Lemma 2.2. When F is convex in R, any viscosity solution of (3.13) is in $C^{1,1}(\mathbb{R}^n)$ (and in fact $C^{2,\gamma}(\mathbb{R}^n)$ for some $\gamma > 0$). The proof of (d) then follows as in Lemma 2.2. □

THEOREM 3.3. *Assume F verifies (3.2)–(3.5). Then u is the unique viscosity solution of*

$$\begin{cases} \bar{F}(D^2u, Du, u, x) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.19}$$

Proof. 1. The proof is similar to that for Theorem 2.3. Fix $\phi \in C^\infty(\Omega)$ and suppose $u - \phi$ has a strict local maximum at $x_0 \in \Omega$, with

$$u(x_0) = \phi(x_0). \tag{3.20}$$

We intend to prove

$$\bar{F}(D^2\phi(x_0), D\phi(x_0), \phi(x_0), x_0) \leqq 0, \tag{3.21}$$

and consequently suppose to the contrary that

$$\bar{F}(D^2\phi(x_0), D\phi(x_0), \phi(x_0), x_0) \equiv \theta > 0. \tag{3.22}$$

Set $R = D^2\phi(x_0)$, $p = D\phi(x_0)$, $u = \phi(x_0)$, $x = x_0$ in (3.13) and choose v to be a viscosity solution of the cell problem

$$\begin{cases} F(D_y^2 v + D^2\phi(x_0), D\phi(x_0), \phi(x_0), x_0, y) \\ \quad = \bar{F}(D^2\phi(x_0), D\phi(x_0), \phi(x_0), x_0) = \theta \quad \text{in } \mathbb{R}^n, \\ v \quad Y\text{-periodic.} \end{cases} \tag{3.23}$$

Define now the perturbed test function

$$\phi^\varepsilon(x) = \phi(x) + \varepsilon^2 v\left(\frac{x}{\varepsilon}\right) \quad (x \in \bar{\Omega}).$$

Note that ϕ^ε is $C^{1,\gamma}$, but is not C^2 in general.

2. We *claim* that if $\varepsilon > 0$ is small enough, then

$$F\left(D^2\phi^\varepsilon(x), D\phi^\varepsilon(x), \phi^\varepsilon(x), x, \frac{x}{\varepsilon}\right) \geqq \frac{\theta}{2} \quad \text{in } B(x_0, r) \tag*{$(3.24)_\varepsilon$}$$

in the viscosity sense, for some sufficiently small $r > 0$.

Select $\psi \in C^\infty(\mathbb{R}^n)$ and suppose $\phi^\varepsilon - \psi$ has a minimum at a point $x_1 \in B(x_0, r)$, with

$$\phi^\varepsilon(x_1) = \psi(x_1). \tag{3.25}$$

Then the mapping

$$y \mapsto v(y) - \eta(y)$$

has a minimum at $y_1 \equiv x_1/\varepsilon$, for

$$\eta(y) \equiv \frac{1}{\varepsilon^2}(\psi(\varepsilon y) - \phi(\varepsilon y)).$$

In as much as v is a viscosity solution of (3.23), we deduce

$$F(D^2\eta(y_1) + D^2\phi(x_0), D\phi(x_0), \phi(x_0), x_0, y_1) \geqq \theta.$$

Thus

$$F\left(D^2\psi(x_1) + D^2\phi(x_0) - D^2\phi(x_1), D\phi(x_0), \phi(x_0), x_0, \frac{x_1}{\varepsilon}\right) \geqq \theta;$$

whence

$$F\left(D^2\psi(x_1), D\phi(x_0), \psi(x_1), x_1, \frac{x_1}{\varepsilon}\right) \geqq \tfrac{3}{4}\theta$$

if $r > 0$ is small enough. In addition, since $v \in C^{1,\gamma}$, we can compute

$$D\psi(x_1) = D\phi^\varepsilon(x_1) = D\phi(x_1) + \varepsilon Dv\left(\frac{x_1}{\varepsilon}\right).$$

Hence

$$F\left(D^2\psi(x_1), D\psi(x_1), \psi(x_1), x_1, \frac{x_1}{\varepsilon}\right) \geqq \frac{\theta}{2}$$

if r, $\varepsilon > 0$ are small enough. This inequality verifies the claim $(3.24)_\varepsilon$.

3. Owing now to $(3.1)_\varepsilon$, $(3.25)_\varepsilon$ and the comparison theorem for viscosity solutions of fully nonlinear elliptic PDE (cf. [**13**, **10**, **12**]), we have

$$(u^\varepsilon - \phi^\varepsilon)(x_0) \leqq \max_{\partial B(x_0, r)} (u^\varepsilon - \phi^\varepsilon).$$

Sending $\varepsilon \to 0$, we discover that

$$(u - \phi)(x_0) \leqslant \max_{\partial B(x_0, r)} (u - \phi),$$

a contradiction since $u - \phi$ has a strict local maximum at x_0. This confirms (3.21).

4. The opposite inequality holds similarly, provided $u - \phi$ has a strict local minimum at a point $x_0 \in \Omega$. $\square$

4. Convergence of second-order to first-order equations

As our next general class of problems, let us investigate the PDE

$$\begin{cases} F\left(\varepsilon D^2 u^\varepsilon, Du^\varepsilon, u^\varepsilon, x, \dfrac{x}{\varepsilon}\right) = 0 & \text{in } \Omega, \\ u^\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1$_\varepsilon$}$$

for

$$F: S^{n\times n} \times \mathbb{R}^n \times \mathbb{R} \times \bar{\Omega} \times \mathbb{R}^n \to \mathbb{R}$$

as above. The new effects concern the interplay between the "vanishing viscosity" term $\varepsilon D^2 u^\varepsilon$ and the high frequency, periodic oscillation term x/ε. We shall assume as usual that

$$\text{the mapping } y \mapsto F(R, p, u, x, y) \text{ is } Y\text{-periodic} \tag{4.2}$$

for all R, p, u, x. Additionally, we make the uniform ellipticity assumption (3.3). We suppose as well that

$$\lim_{|p|\to\infty} F(0, p, u, x, y) = +\infty \text{ uniformly on } B(0, L) \times \bar{\Omega} \times \mathbb{R}^n \text{ for each } L > 0 \tag{4.3}$$

$$u \mapsto F(R, p, u, x, y) - \mu u \text{ is nondecreasing} \tag{4.4}$$

for some $\mu > 0$ and all R, p, x, y. Finally, we require

$$F \text{ is Lipschitz on } S^{n\times n} \times \mathbb{R}^n \times \mathbb{R} \times \bar{\Omega} \times \mathbb{R}^n. \tag{4.5}$$

Now, in the light of hypotheses (3.3) and (4.4), we have the estimate

$$\sup_{0<\varepsilon<1} \|u^\varepsilon\|_{L^\infty(\Omega)} \leqq \frac{1}{\mu} \|F(0, 0, 0, \cdot, \cdot)\|_{L^\infty(\bar{\Omega}\times Y)} < \infty. \tag{4.6}$$

In view of the ε in front of the second derivatives in $(4.1)_\varepsilon$, we do not have available Hölder estimates as in Section 3. Our plan, as before, is to show that the u^ε converge uniformly to a viscosity solution u of an effective limit PDE. The new complication is that we cannot at once legitimately deduce uniform convergence of any subsequence from the crude bound (4.6). We instead utilise the techniques of Ishii [**11**] and Barles and Perthame [**1**].

LEMMA 4.1. *For each fixed $p \in \mathbb{R}^n$, $u \in \mathbb{R}$ and $x \in \bar{\Omega}$, there exists a unique real number λ for which the PDE*

$$\begin{cases} F(D_y^2 v, D_y v + p, u, x, y) = \lambda & \text{in } \mathbb{R}^n, \\ v \quad Y\text{-periodic}, \end{cases} \tag{4.7}$$

has a viscosity solution $v \in C^{1,\gamma}(\mathbb{R}^n)$ for some $\gamma > 0$.

Proof. The proof is similar to that for Lemma 3.1. For each $\delta > 0$ we solve

$$\delta w^\delta + F(D_y^2 w^\delta, D_y w^\delta + p, u, x, y) = 0 \text{ in } \mathbb{R}^n \tag*{$(4.8)_\delta$}$$

and then show

$$\begin{cases} \delta w^\delta \to -\lambda \\ v^\delta \to v \end{cases} \quad \text{uniformly on } \mathbb{R}^n \tag{4.9}$$

for

$$v^\delta \equiv w^\delta - \min_Y w^\delta. \qquad \square$$

We display the dependence of λ on p, u, x by writing

$$\lambda = \bar{F}(p, u, x). \tag{4.10}$$

Equation (4.7) thus becomes

$$\begin{cases} F(D_y^2 v, D_y v + p, u, x, y) = \bar{F}(p, u, x) \text{ in } \mathbb{R}^n, \\ v \quad Y\text{-periodic}. \end{cases} \tag{4.11}$$

LEMMA 4.2. (a) $\lim_{|p| \to \infty} \bar{F}(p, u, x) = +\infty$, *uniformly on $B(0, L) \times \bar{\Omega}$ for each $L > 0$.*

(b) *The mapping $u \mapsto \bar{F}(p, u, x) - \mu u$ is nondecreasing for all p, x.*

(c) *$\bar{F}$ is Lipschitz on $\mathbb{R}^n \times \mathbb{R} \times \bar{\Omega}$.*

(d) *If F is convex in R and p, $\bar{F}$ is convex in p.*

Proof. 1. Fix $M > 0$. We consider $(4.8)_\delta$ and evaluate at a point $y_0 \in Y$ where w^δ attains its maximum. Thus

$$\delta w^\delta(y_0) + F(0, p, u, x, y_0) \leq 0.$$

Hypothesis (4.3) now implies

$$-\delta w^\delta(y_0) \geqq M$$

provided $|p|$ is large enough. Assertion (a) then follows from (4.9), (4.10).

2. Assertions (b)–(d) are proved as in Lemmas 2.2, 3.2. $\square$

LEMMA 4.3. *There exists a unique viscosity solution $u \in C^{0,1}(\bar{\Omega})$ of*

$$\begin{cases} \bar{F}(Du, u, x) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.12}$$

Proof. This is a consequence of conditions (a)–(c) in Lemma 4.2. $\square$

THEOREM 4.4. *Suppose F satisfies* (3.3), (4.2)–(4.5). *Then*

$$u^\varepsilon \to u \quad \textit{uniformly on } \bar{\Omega},$$

where u is the unique viscosity solution of (4.12).

Proof. 1. Set

$$u^*(x) \equiv \limsup_{\substack{\varepsilon\to 0\\ z\to x}} u^\varepsilon(z) \quad (x \in \bar{\Omega})$$

$$u_*(x) \equiv \liminf_{\substack{\varepsilon\to 0\\ z\to x}} u^\varepsilon(z) \quad (x \in \bar{\Omega}).$$

Then u^* is upper semicontinuous, u_* is lower semicontinuous,

$$u_* \leqq u^* \quad \text{in } \Omega. \tag{4.13}$$

Our intention is ultimately to show

$$u^* = u_* = u, \tag{4.14}$$

u the (unique) viscosity solution of (4.12).

2. We first assert that

$$\bar{F}(Du^*, u^*, x) \leqq 0 \quad \text{in } \Omega \tag{4.15}$$

in the viscosity sense. To prove this, select $\phi \in C^\infty(\mathbb{R}^n)$ and suppose $u^* - \phi$ has a strict local maximum at $x_0 \in \Omega$, with

$$u^*(x_0) = \phi(x_0). \tag{4.16}$$

We wish to show that

$$\bar{F}(D\phi(x_0), \phi(x_0), x_0) \leqq 0 \tag{4.17}$$

and consequently assume to the contrary that

$$\bar{F}(D\phi(x_0), \phi(x_0), x_0) \equiv \theta > 0. \tag{4.18}$$

Set $p = D\phi(x_0)$, $u = \phi(x_0)$, $x = x_0$ in (4.11).

Let v be a viscosity solution of (4.11), and define the perturbed test function

$$\phi^\varepsilon(x) \equiv \phi(x) + \varepsilon v\left(\frac{x}{\varepsilon}\right) \quad (x \in \bar{\Omega}).$$

3. We *claim* that if $\varepsilon > 0$ is small enough, then

$$F\left(\varepsilon D^2\phi^\varepsilon(x), D\phi^\varepsilon(x), \phi^\varepsilon(x), x, \frac{x}{\varepsilon}\right) \geqq \frac{\theta}{2} \quad \text{in } B(x_0, r) \tag{4.19}$$

in the viscosity sense, for some sufficiently small $r > 0$. To see this, fix $\psi \in C^\infty(\mathbb{R}^n)$ and suppose $\psi^\varepsilon - \psi$ has a minimum at a point $x_1 \in B(x_0, r)$, with

$$\phi^\varepsilon(x_1) = \psi(x_1).$$

Then

$$y \mapsto v(y) - \eta(y)$$

has a minimum at $y_1 = x_1/\varepsilon$, for

$$\eta(y) \equiv \frac{1}{\varepsilon}(\psi(\varepsilon y) - \phi(\varepsilon y)) \quad (y \in \mathbb{R}^n).$$

Since v is a viscosity solution of (4.11), we have

$$F(D^2\eta(y_1), D\eta(y_1) + D\phi(x_0), \phi(x_0), x_0, y_1) \geqq \theta.$$

Hence

$$F\left(\varepsilon D^2\psi(x_1) - \varepsilon D^2\phi(x_1), D\psi(x_1) + D\phi(x_0) - D\phi(x_1), \phi(x_0), x_0, \frac{x_1}{\varepsilon}\right) \geqq \theta;$$

and so

$$F\left(\varepsilon D^2\psi(x_1), D\psi(x_1), \psi(x_1), x_1, \frac{x_1}{\varepsilon}\right) \geqq \frac{\theta}{2}$$

provided ε, $r > 0$ are small enough. This verifies (4.19).

4. Since u^ε solves $(4.1)_\varepsilon$, we deduce

$$\max_{B(x_0,r)} (u^\varepsilon - \phi) \leqq \max_{\partial B(x_0,r)} (u^\varepsilon - \phi^\varepsilon),$$

whence we arrive after sending $\varepsilon \to 0$ at the contradiction

$$(u^* - \phi)(x_0) \leqq \max_{\partial B(x_0,r)} (u^* - \phi).$$

5. We next assert

$$u^* = 0 \quad \text{on } \partial\Omega. \tag{4.20}$$

Indeed, by definition and $(4.1)_\varepsilon$, we see

$$u^* \geqq 0 \quad \text{on } \partial\Omega. \tag{4.21}$$

To establish the opposite inequality, fix any point $x_0 \in \partial\Omega$ and choose a smooth function v such that

$$v(x_0) = 0, \quad v > 0 \text{ in } \bar{\Omega} - \{x_0\}, \quad Dv \neq 0 \text{ on } \bar{\Omega}. \tag{4.22}$$

Fix $\alpha > 0$ so large that

$$F\left(0, \alpha Dv, 0, x, \frac{x}{\varepsilon}\right) \geqq 1 \tag{4.23}$$

for each $x \in \Omega$, $0 < \varepsilon < 1$; this is possible owing to (4.3), (4.22). Then

$$\begin{aligned} F\left(\varepsilon\alpha D^2 v, \alpha Dv, \alpha v, x, \frac{x}{\varepsilon}\right) &\geqq F\left(\varepsilon\alpha D^2 v, \alpha Dv, 0, x, \frac{x}{\varepsilon}\right) \quad \text{by (4.4), (4.22)} \\ &\geqq \tfrac{1}{2} \quad \text{on } \bar{\Omega} \quad \text{by (4.23)}, \end{aligned}$$

provided $\varepsilon > 0$ is small enough. By comparison therefore

$$u^\varepsilon \leqq \alpha v \quad \text{on } \bar{\Omega}.$$

Consequently

$$u^* \leqq \alpha v \quad \text{on } \bar{\Omega},$$

and thus

$$u^*(x_0) \leqq 0.$$

As this inequality is valid for each point $x_0 \in \partial\Omega$, we deduce

$$u^* \leqq 0 \quad \text{on } \partial\Omega.$$

This and (4.21) establish (4.20).

6. Using now (4.15), (4.20), we deduce

$$u^* \leqq u \quad \text{in } \Omega, \tag{4.24}$$

u the unique viscosity solution of (4.12). An analogous argument reveals

$$u \leqq u_* \quad \text{in } \Omega.$$

This estimate, (4.24), and (4.13) lead us at last to (4.14). □

5. Some examples and variants

It remains a fascinating and still largely open problem to discover explicit formulae for the effective nonlinearities $\bar{H}$ and $\bar{F}$ as above. The following are some partial results in this direction. See also Lions, Papanicolaou and Varadhan [**14**].

1. *A quadratic Hamiltonian.* Consider the problem

$$\begin{cases} F\left(\varepsilon D^2u^\varepsilon, Du^\varepsilon, u^\varepsilon, x, \dfrac{x}{\varepsilon}\right) \equiv \mu u^\varepsilon - \varepsilon a_{ij}\left(\dfrac{x}{\varepsilon}\right)u^\varepsilon_{x_ix_j} + a_{ij}\left(\dfrac{x}{\varepsilon}\right)u^\varepsilon_{x_i}u^\varepsilon_{x_j} - f = 0 \quad \text{in } \Omega, \\ u^\varepsilon = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{5.1$_\varepsilon$}$$

where the smooth symmetric coefficients $\{a_{ij}\}_{i,j=1}^n$ satisfy

$$y \mapsto a_{ij}(y) \text{ is } Y\text{-periodic } (1 \leqq i, j \leqq n) \tag{5.2}$$

and

$$\begin{cases} \text{there exists } \theta > 0 \text{ such that} \\ \theta\,|\xi|^2 \leqq a_{ij}(y)\xi_i\xi_j \quad (\xi \in \mathbb{R}^n) \end{cases} \tag{5.3}$$

for all $y \in \mathbb{R}^n$. (This example does not verify hypothesis (4.5) because of the quadratic growth in the gradient, but this fact does not affect the following explicit analysis.) According to Section 4, the corresponding cell problem is

$$\begin{cases} \mu u - a_{ij}(y)v_{y_iy_j} + a_{ij}(y)(v_{y_i} + p_i)(v_{y_j} + p_j) - f(x) = \lambda \quad \text{in } Y, \\ v \quad Y\text{-periodic.} \end{cases} \tag{5.4}$$

Assume temporarily v to be a smooth solution of (5.4) and set

$$w \equiv e^{-v}. \tag{5.5}$$

Then w solves the *linear* eigenvalue problem

$$\begin{cases} L^p w = (\mu u - (\lambda + f(x)))w & \text{in } \mathbb{R}^n, \\ w \quad Y\text{-periodic} \end{cases} \tag{5.6}$$

for

$$L^p w \equiv -a_{ij}(y) w_{y_i y_j} + 2a_{ij}(y) p_i w_{y_j} - a_{ij}(y) p_i p_j w. \tag{5.7}$$

As $w > 0$, we see from (5.6) and the Krein–Rutman Theorem that $\mu u - (\lambda + f(x))$ is the principle eigenvalue $\lambda^0(p)$ of the operator L^p (with periodic boundary conditions on Y). Hence

$$\lambda = \bar{F}(p, u, x) = \mu u - \lambda^0(p) - f(x).$$

Thus as in Section 4

$$u^\varepsilon \to u \quad \text{uniformly on } \bar{\Omega},$$

u the unique viscosity solution of

$$\begin{cases} \mu u - \lambda^0(Du) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Consider next the first order PDE

$$\begin{cases} H\left(Du^\varepsilon, u^\varepsilon, x, \dfrac{x}{\varepsilon}\right) = \mu u^\varepsilon + a_{ij}\left(\dfrac{x}{\varepsilon}\right) u^\varepsilon_{x_i} u^\varepsilon_{x_j} - f = 0 & \text{in } \Omega \\ u^\varepsilon = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{5.8$_\varepsilon$}$$

the coefficients $\{a_{ij}\}_{i,j=1}^n$ verifying (5.2), (5.3). According to Section 2, the cell problem is

$$\begin{cases} \mu u + a_{ij}(y)(v_{y_i} + p_i)(v_{y_j} + p_j) - f(x) = \lambda & \text{in } \mathbb{R}^n, \\ v \quad Y\text{-periodic}. \end{cases} \tag{5.9}$$

To solve this, let us fix $\delta > 0$ and study the approximating problem

$$\begin{cases} \mu u - \delta^2 a_{ij}(y) v^\delta_{y_i y_j} + a_{ij}(y)(v^\delta_{y_i} + p_i)(v^\delta_{y_j} + p_j) - f(x) = \lambda^\delta & \text{in } \mathbb{R}^n, \\ v^\delta \quad Y\text{-periodic}. \end{cases} \tag{5.10$_\delta$}$$

Setting

$$w^\delta \equiv e^{-v^\delta/\delta},$$

we transform $(5.10)_\delta$ into the linear eigenvalue problem

$$\begin{cases} L^{p/\delta} w^\delta = \dfrac{\mu u - (\lambda^\delta + f(x))}{\delta^2} w^\delta & \text{in } \mathbb{R}^n, \\ w^\delta \quad Y\text{-periodic}. \end{cases} \tag{5.11$_\delta$}$$

Since $w^\delta > 0$, we deduce as above that $[\mu u - (\lambda^\delta + f(x))]/\delta^2$ is the principle eigenvalue $\lambda^0(p/\delta)$ of the operator $L^{p/\delta}$. Therefore

$$\lambda^\delta = \mu u - \delta^2 \lambda^0\left(\frac{p}{\delta}\right) - f(x) \quad (\delta > 0). \tag{5.12}$$

It is easy to verify using (5.3) that

$$a|p|^2 - b \leq -\lambda^0(p) \leq A|p|^2 + B \quad (p \in \mathbb{R}^n),$$

for appropriate constants A, B, a, $b > 0$. Thus for fixed p, u, x

$$\sup_{0<\delta<1} |\lambda^\delta| < \infty.$$

In consequence we deduce from $(5.10)_\delta$ the bounds

$$\sup_{0<\delta<1} \|v^\delta\|_{C^{0,1}(\mathbb{R}^n)} < \infty$$

provided we add as necessary a constant to v^δ to ensure

$$\min v^\delta = 0.$$

Choose then a subsequence $\{(v^{\delta_j}, \lambda^{\delta_j})\}_{j=1}^\infty \subset \{(v^\delta, \lambda^\delta)\}_{\delta>0}$,

$$\begin{cases} \delta_j \to 0 \\ \lambda^{\delta_j} \to \lambda \quad \text{uniformly on } \mathbb{R}^n. \\ v^{\delta_j} \to v \end{cases}$$

It follows that v and λ solve (5.9) in the viscosity sense. In view of the uniqueness of λ (Lemma 2.1) we deduce that in fact

$$\lim_{\delta\to 0^+} \lambda^\delta = \lambda.$$

Thus the full limit

$$\lambda^*(p) \equiv \lim_{\delta\to 0^+} \delta^2 \lambda^0\left(\frac{p}{\delta}\right) \tag{5.13}$$

exists for each $p \in \mathbb{R}^n$. We verify also that λ^* is concave and homogeneous of degree two. Finally we see that the Hamiltonian is

$$\bar{H}(p, x) = -\lambda^*(p) - f(x).$$

Theorem 2.3 now asserts that

$$u^\varepsilon \to u \quad \text{uniformly on } \bar{\Omega},$$

u the unique viscosity solution of

$$\begin{cases} \mu u - \lambda^*(Du) = f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{5.14}$$

I do not know any explicit characterisation of the function $\lambda^*(\cdot)$ beyond the representation formula (5.13).

2. *A scalar conservation law with periodic forcing.* As a rather different example, we consider next a scalar conservation law driven by a large amplitude, rapidly oscillating, time-periodic forcing term of mean zero. The relevant PDE is

$$\begin{cases} u_t^\varepsilon + G(u^\varepsilon)_x = \dfrac{1}{\varepsilon} f\left(x, \dfrac{t}{\varepsilon}\right) \quad \text{in } \mathbb{R} \times (0, \infty), \\ u^\varepsilon = g \quad \text{on } \mathbb{R} \times \{t = 0\}. \end{cases} \tag{5.15_ε}$$

We assume $g \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and also that the smooth function G satisfies

$$\lim_{|u| \to \infty} G(u) = \infty. \tag{5.16}$$

We additionally assume f is smooth with

$$s \mapsto f(x, s) \quad \text{is 1-periodic } (x \in \mathbb{R}), \tag{5.17}$$

$$\int_0^1 f(x, s)\, ds = 0 \quad (x \in \mathbb{R}). \tag{5.18}$$

We inquire, as usual, as to the limit of u^ε as $\varepsilon \to 0$. For this, set

$$F(x, s) \equiv \int_0^s f(x, t)\, dt \quad (x, s \in \mathbb{R}),$$

so that

$$s \mapsto F(x, s) \quad \text{is} \quad \text{1-periodic}$$

and

$$F(x, 0) = F(x, 1) = 0 \quad (x \in \mathbb{R}).$$

Consider the Hamilton-Jacobi PDE

$$\begin{cases} w_t^\varepsilon + H\left(w_x^\varepsilon, x, \dfrac{t}{\varepsilon}\right) = 0 & \text{in } \mathbb{R} \times (0, \infty), \\ w^\varepsilon = h & \text{on } \mathbb{R} \times \{t = 0\}, \end{cases} \tag*{$(5.19)_\varepsilon$}$$

where

$$h(x) = \int_{-\infty}^x g(y)\, dy \quad (x \in \mathbb{R})$$

and

$$H(p, x, s) \equiv G(p + F(x, s)) \quad (p, x, s \in \mathbb{R}). \tag{5.20}$$

Owing to (5.16), for each $\varepsilon > 0$ the PDE $(5.19)_\varepsilon$ possesses a unique viscosity solution w^ε, with the estimates

$$\sup_{0<\varepsilon<1} \|w^\varepsilon, w_x^\varepsilon, w_t^\varepsilon\|_{L^\infty(\mathbb{R} \times (0,\infty))} < \infty.$$

As w^ε is Lipschitz,

$$u^\varepsilon(x, t) \equiv w_x^\varepsilon(x, t) + F\left(x, \frac{t}{\varepsilon}\right) \tag{5.21}$$

exists for almost every (x, t). In addition, u^ε so defined is the unique entropy solution of $(5.15)_\varepsilon$.

Pass to a subsequence $\{w^{\varepsilon_j}\}_{j=1}^\infty \subset \{w^\varepsilon\}_{\varepsilon>0}$ with

$$w^{\varepsilon_j} \to w \quad \text{locally uniformly in } \mathbb{R} \times [0, \infty) \tag{5.22}$$

for $w \in C^{0,1}(\mathbb{R} \times [0, \infty))$. To find the effective PDE which w solves, let us introduce as in Sections 2–4 an appropriate cell problem, which is, for the case at hand,

$$\begin{cases} v_s + H(D_y v + p, x, s) = \lambda & \text{in } \mathbb{R} \times [0, \infty), \\ v \quad \text{1-periodic in } s, \end{cases} \tag{5.23}$$

for $p, x, y \in \mathbb{R}$. We solve (5.23) by seeking a solution $v(s)$ which does not depend on y. Thus (5.23) becomes the ODE

$$\begin{cases} v_s + H(p, x, s) = \lambda & \text{in } \mathbb{R}, \\ v \quad \text{1-periodic in } s. \end{cases} \tag{5.24}$$

Setting

$$v(0) = 0,$$

we compute

$$v(1) = \int_0^1 v_s(s)\, ds = \lambda - \int_0^1 H(p, x, s)\, ds = 0,$$

provided

$$\lambda = \bar{H}(p, x) \equiv \int_0^1 G(p + F(x, s))\, ds \quad (p, x \in \mathbb{R}^n).$$

As in Section 2, we can show that w is the unique viscosity solution of

$$\begin{cases} w_t + \bar{H}(w_x, x) = 0 & \text{in } \mathbb{R} \times [0, \infty), \\ w = h & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases} \tag{5.25}$$

Now according to (5.21), (5.22),

$$u^\varepsilon(x, t) \overset{*}{\rightharpoonup} w_x(x, t) + \bar{F}(x) \equiv u$$

weakly-star in $L^\infty(\mathbb{R} \times [0, \infty))$, for

$$\bar{F}(x) \equiv \int_0^1 F(x, s)\, ds = \int_0^1 \left(\int_0^s f(x, t)\, dt \right) ds.$$

In view of (5.25),

$$\tilde{u} \equiv w_x$$

is the unique entropy solution of the conservation law

$$\begin{cases} \tilde{u}_t + \bar{H}(\tilde{u}, x)_x = 0 & \text{in } \mathbb{R} \times [0, \infty) \\ \tilde{u} = g & \text{on } \mathbb{R} \times [0, \infty). \end{cases}$$

Hence u is the unique entropy solution of

$$\begin{cases} u_t + \bar{G}(u, x)_x = 0 & \text{in } \mathbb{R} \times [0, \infty) \\ u = \bar{g} & \text{on } \mathbb{R} \times [0, \infty), \end{cases}$$

where

$$\bar{G}(u, x) = \bar{H}(u - \bar{F}(x), x) = \int_0^1 G(u + F(x, s) - \bar{F}(x))\, ds \quad (u, x \in \mathbb{R})$$

and

$$\bar{g} \equiv g + \bar{F}.$$

Notice in this example that the initial conditions change in the limit.

See [**8**] for homogenisation of a conservation law with a rapidly oscillating space-periodic forcing term.

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